Over-coming the fluid-structure added-mass instability for incompressible flows

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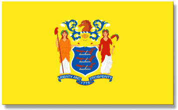
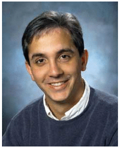


Acknowledgments.

Primary collaborators:

Jeff Banks
(RPI)



Kyle Chand
(LLNL)

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(RPI)

Other contributors to FSI work:

1. Dr. Björn Sjögreen (LLNL).
2. Dr. Alex Main (Post-doc, Duke)

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Outline

1. Introduction:
   1. The Overture framework for solving PDEs.
   2. Overlapping grids – why are they useful?
   3. Composite Grid PDE solvers and applications.
2. Fluid-structure interaction (FSI) algorithms
   1. Our FSI approach using deforming composite grids.
   2. Traditional partitioned FSI algortihms.
   3. Added-mass instabilities.
3. Added-mass partitioned (AMP) schemes
   1. Compressible flows + rigid/deforming solids
   2. Incompressible flows + beams/bulk-solids

The Overture project is developing PDE solvers for a wide class of continuum mechanics applications.

Overture is a toolkit for solving PDE's on overlapping grids and includes CAD, grid generation, numerical approximations, AMR and graphics.

The CG (Composite Grid) suite of PDE solvers (cgcns, cgins, cgmx, cgsm, cgad, cgmp) provide algorithms for modeling gases, fluids, solids and E&M.

Overture and CG are open source and available from overtureFramework.org.

We are looking at a variety of applications:

- wind turbines, building flows (cgins),
- explosives modeling (cgcns),
- fluid-structure interactions (e.g. blast effects) (cgmp+cgcns+cgsm),
- conjugate heat transfer (e.g. NIF holhraum) (cgmp+cgins+cgad),
- damage mitigation in NIF laser optics (cgmx).

Overture simulations from the Composite Grid (CG) solver suite.

Cgins: incompressible flow

Cgins

Cgsm: elasticity

Cgsm

Cgcns: compressible flow

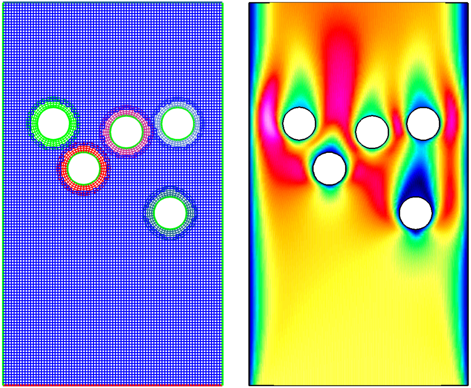
Cgcns

Cgmx: electromagnetics

Cgmx

What are overlapping grids and why are they useful?

Overlapping grid: a set of structured grids that overlap.



- Overlapping grids can be rapidly generated as bodies move.
- High quality grids under large displacements.
- Cartesian grids for efficiency.
- Smooth grids for accuracy at boundaries.
- Efficient for high-order methods.

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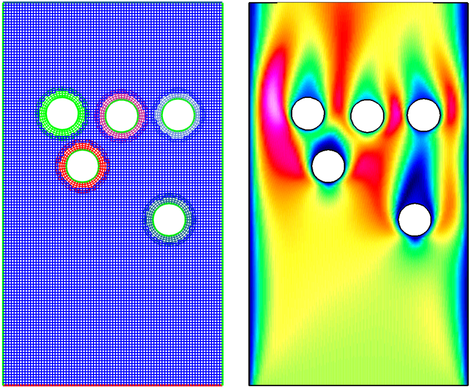

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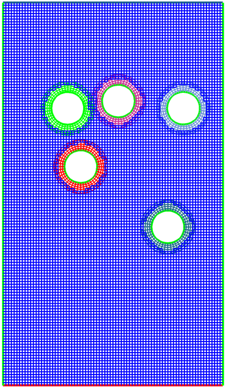

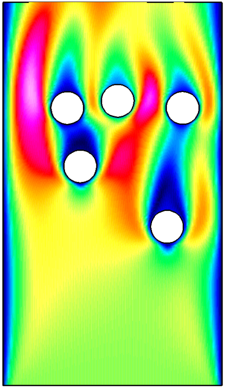

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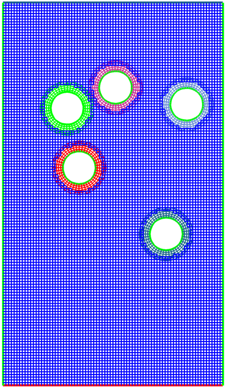

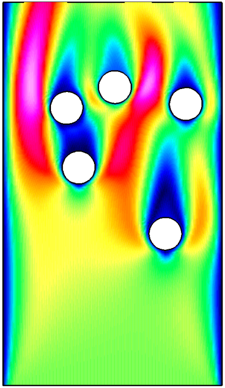

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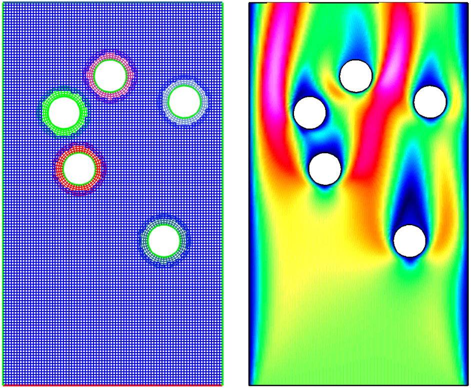

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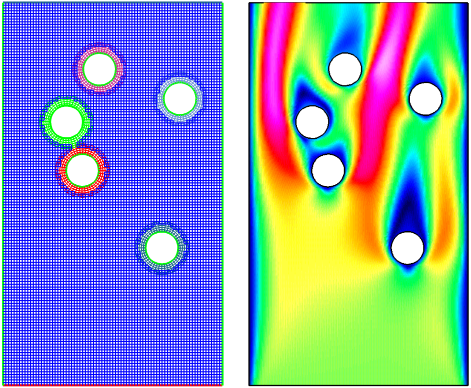

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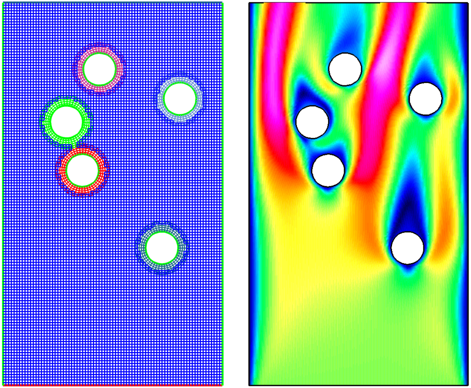

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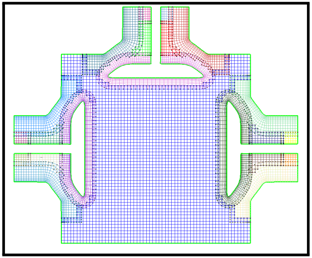
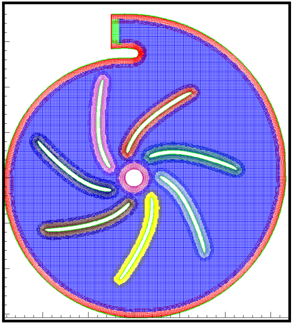
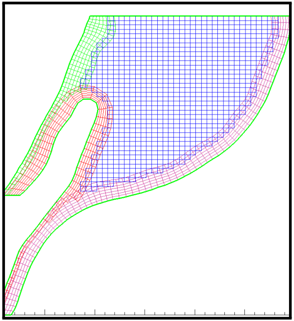
Asymptotic Performance Principle for overlapping grids

As grids are refined, total CPU/memory usage can approach that of a Cartesian grid.

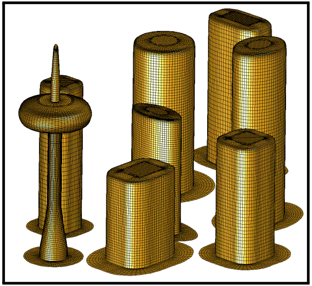
Sample 2D overlapping grids (Ogen).

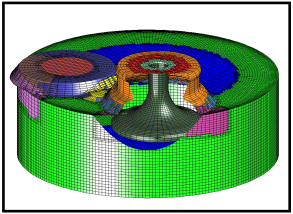


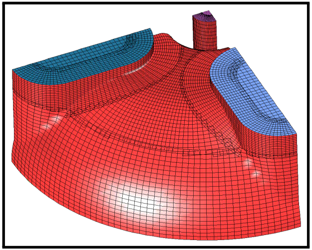


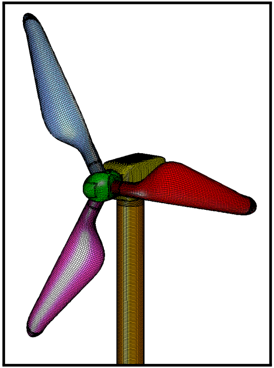


Sample 3D overlapping grids (Ogen).







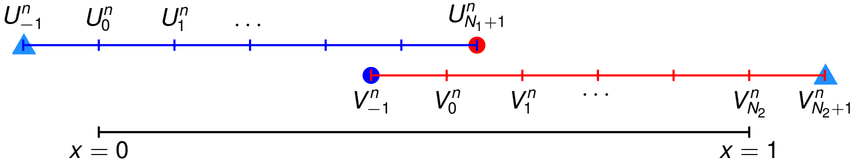


A one-dimensional overlapping grid example

To solve the advection-diffusion equation

$$\begin{aligned} u_t + a u_x &= \nu u_{xx}, && x \in (0,1) \\ u(0,t) = g_0(t), \quad u_x(1,t) &= g_1(t), && \text{(boundary conditions)} \\ u(x,0) &= u_0(x), && \text{(initial conditions)} \end{aligned}$$

introduce the grid functions $U_i^n \approx u(x_i^{(1)}, n\Delta t)$, $V_i^n \approx u(x_i^{(2)}, n\Delta t)$, and the overlapping grid:



How to advance the solution on an overlapping grid.

(1) interior equations, (2) boundary conditions, (3) interpolation points.

Given the solution at time t^n, compute the solution at time t^{n+1}:

$$(U_i^{n+1} - U_i^n)/\Delta t = -a\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x} + \nu\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}, \qquad i = 1, 2, \ldots, N_1$$

$$(V_j^{n+1} - V_j^n)/\Delta t = -a\frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x} + \nu\frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{\Delta x^2}, \qquad j = 0, 2, \ldots, N_2$$

$$U_0^{n+1} = g(t^n), \quad D_0 V_{N_2}^{n+1} = g_1(t^{n+1}), \qquad \text{(boundary conditions)}$$

$$U_{N_1+1}^{n+1} = (1-\alpha)(1-\frac{\alpha}{2})\ V_{-1}^{n+1} + \alpha(2-\alpha)\ V_0^{n+1} + \frac{\alpha}{2}(\alpha-1)\ V_1^{n+1}, \quad \text{(interpolation)}$$

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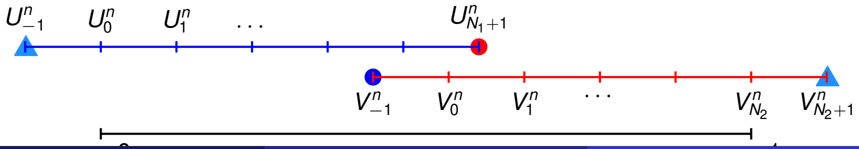

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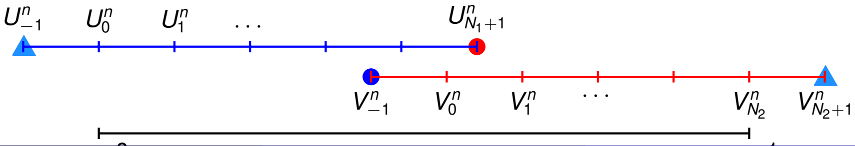

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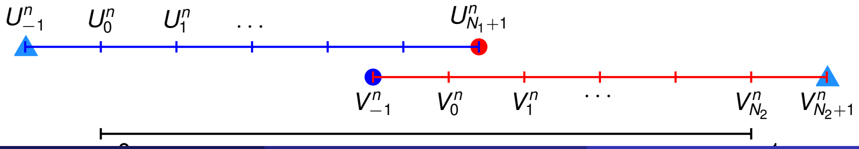
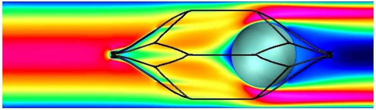
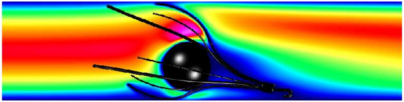


Overture is used by research groups worldwide

Typical users are graduate students and University/Lab researchers.

Collaborations
Wind turbines (Prof. Ravi Samtaney, KAUST, Dr. Stephen Guzik CSU, Prof. Xinfeng Gao CSU, Prof. Dale Pullin, Caltech,).
Tear films and droplets (Dr. Kara Maki, RIT, and Prof. Richard Braun, U. Delaware).
Electromagnetics for DOD applications (Dr. Alex Fedoseyev, Computational Sciences LLC)
Centrifugal pumps (Dr. Franck Monmont, Schlumberger Research)
Reduced order models and control (Prof. John Burns, Prof. Jeff Borggaard, Virginia Tech).
Flapping airfoils, mircro-air vehicles (Prof. Yongsheng Lian, U. of Louisville).
Water waves (Prof. Harry Bingham, Technical U. of Denmark).
Plasma physics (Dr. Jeff Banks RPI, Dr. Richard Berger, LLNL).
Blood flow and blood clot filters (Dr. Mike Singer).
Flapping airfoils (Dr. Joel Guerrero, U. of Genoa).
High-order accurate subsonic/transonic aero-acoustics (Dr. Philippe Lafon, CNRS, EDF).
Elastic wave equation (Dr. Daniel Appelö, U. of New Mexico).
Relativistic hydrodynamics and Einstein field equations (Dr. Philip Blakely, U. Cambridge).
Converging shock waves, shock focusing (Prof. Veronica Eliasson, USC).
Pitching airfoils (Dr. D. Chandar, U. of Wyoming, Prof. M. Damodaran, NTU, Singapore).
Hypersonic flows for reentry vehicles, (Dr. Bjorn Sjögreen, LLNL, Dr. Helen Yee, NASA).
High-order accurate, compact Hermite-Taylor schemes (Prof. Tom Hagstrom, SMU).
Aerodynamics (Prof. Daniel Bodony, U. Illinois).



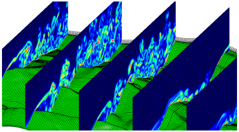


Blood flow past wire frame clot filters. M. Singer et.al.

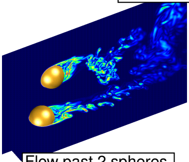
Area: Incompressible flow

Software: Cgins: efficient fractional-step code for the incompressible Navier-Stokes equations on moving overlapping grids.
Current work: Parallel and high-order accurate approximate-factored scheme, matrix-free multigrid solver, moving grid generation.
Current applications: wind farms, wind turbines, micro-air vehicles, blood flow, centrigual pumps.

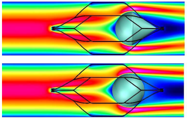
Incompressible flows.



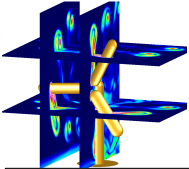
Flow over terrain.



Flow past 2 spheres.



Blood flow past clot filter.



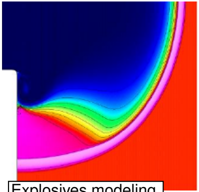
Flow past rotating blades.

- T.M. Broering, Y. Lian, WDH, *Numerical Investigation of Energy Extraction in a Tandem Flapping Wing Configuration*, AIAA J., 2012.
- M. A. Singer, WDH, S. L. Wang, *Computational Modeling of Blood Flow in the Trapease Inferior Vena Cava Filter*, J. Vascular and Interventional Radiology, 2009,
- WDH, *On Multigrid For Overlapping Grids*, SISC, 2005.
- WDH, N.A. Petersson, *A Split-Step Scheme for the Incompressible Navier-Stokes Equations*, 2003.
- WDH., *A Fourth-Order Accurate Method for the Incompressible Navier-Stokes Equations on Overlapping Grids*, J. Comput. Phys, 1994.

Area: High-speed compressible and reactive flows

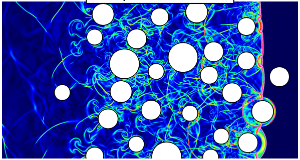
Software: Cgcns: upwind schemes for reactive single-phase, multi-phase and multi-fluid flows on moving and adaptive overlapping grids.

Applications: Explosives modeling, blasts effects, converging and diffracting shocks.

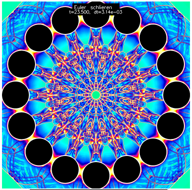


Explosives modeling.

Compressible flows.



Detonation hitting moving cylinders.



Converging shock.

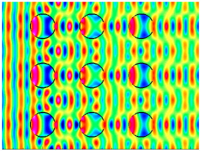
- M. Ozlem, D.W. Schwendeman, A.K. Kapila, WDH, *A numerical study of shock-induced cavity collapse*, Shock Waves, 2012.
- DWS, AKK, WDH, *A Hybrid Two-Phase Mixture Model of Detonation Diffraction with Compliant Confinement*, Comptes Rendus Mathematique, 2012.
- J.W. Banks, WDH, DWS, AKK, *A Study of Detonation Propagation and Diffraction with Compliant Confinement*, Combus. Theory and Modeling, 2008.
- WDH., D. W. Schwendeman, *Parallel Computation of Three-Dimensional Flows using Overlapping Grids with Adaptive Mesh Refinement*, J. Comp. Phys. (2008).
- WDH., DWS, *Moving Overlapping Grids with Adaptive Mesh Refinement for High-Speed Reactive and Nonreactive Flow*, J. Comp. Phys. (2005).
- WDH., DWS, An adaptive numerical scheme for high-speed reactive flow on overlapping grids, J. Comp. Phys. (2003).

Area: Wave propagation

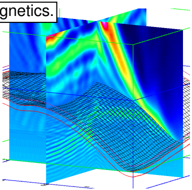
Electromagnetics (EM) and Solid Mechanics (SM).

Software: Cgmx: fourth-order accurate solver for Maxwell's equations.
Software: Cgsm: elastic wave equation solver: first-order system upwind scheme, second-order system scheme.
Applications: EM: accelerator wake fields, laser damage mitigation for optics. SM: fluid-structure interactions.
New directions: EM: upwind schemes for second-order systems. SM: nonlinear solids.

Electromagnetics.

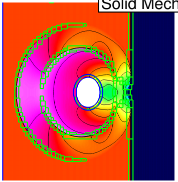


EM scattering from dielectric cylinders.

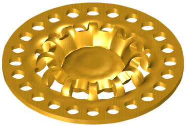


EM intensity at a glass-air interface.

Solid Mechanics.



Diffraction of a p-wave *shock* from a cylindrical void.



Elastic waves in a 3D plate.

- D. Appelö, J.W. Banks , WDH, D.W. Schwendeman, *Numerical Methods for Solid Mechanics on Overlapping Grids: Linear Elasticity*, JCP, 2012.
- WDH., *A High-Order Accurate Parallel Solver for Maxwell's Equations on Overlapping Grids*, SIAM J. Sci. Comput., (2006).

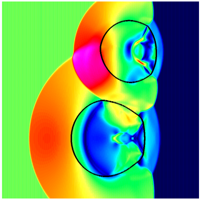
Area: Fluid-structure interactions (FSI).

Software: Cgmp: multi-domain, multi-physics solver that couples the Cgins, Cgcns, Cgad, and Cgsm solvers.

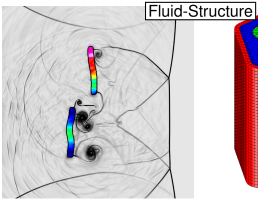
Algorithms: Deforming Composite Grid (DCG) algorithms. Stable partitioned algorithms for FSI and compressible flows. Accurate partitioned algorithms for conjugate heat transfer and incompressible flows.

Applications: Blast effects, explosives, NIF targets, nuclear reactors.

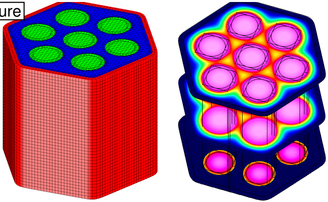
New directions: FSI for incompressible flows.



Shock impacting two elastic cylinders.



Shock impacting two elastic sticks.



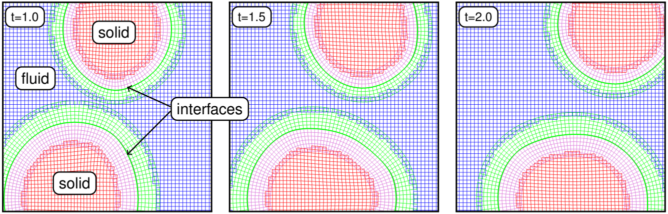
Conjugate heat transfer in a reactor fuel assembly.

- J.W. Banks, WDH, B. Sjögreen, *A stable FSI algorithm for light rigid bodies in compressible flow*, JCP, (accepted).
- J.W. Banks, WDH, D.W. Schwendeman, *Deforming Composite Grids for Solving Fluid Structure Problems*, JCP, 2012.
- WDH., K. K. Chand, *A Composite Grid Solver for Conjugate Heat Transfer in Fluid-Structure Systems*, JCP, 2009.

Our approach in Overture is based on deforming composite grids.

Approach:

- Fluids are solved in a mixed Eulerian-Lagrangian frame.
- Solids are solved in a Lagrangian frame.
- Deforming interface grids are regenerated at each time-step with the hyperbolic grid generator.



procedure SOLVEFSIDCG($\mathcal{G}$, t_{final}) ▷ Input: initial composite grid and final time

  $t := 0$; $n := 0$; $\mathcal{G}^n = \mathcal{G}$;

  assignInitialConditions($\mathbf{q}_{\mathbf{i}}^n$, $\bar{\mathbf{q}}_{\mathbf{i}}^n$, $\mathcal{G}^n$);

  while $t < t_{\text{final}}$ **do**

    $\Delta t :=$ computeTimeStep($\mathbf{q}_{\mathbf{i}}^n$, $\bar{\mathbf{q}}_{\mathbf{i}}^n$, $\mathcal{G}^n$);

    $\mathcal{G}^p :=$ moveGrids($\mathcal{G}^n$, $\mathbf{q}_{\mathbf{i}}^n$, $\bar{\mathbf{q}}_{\mathbf{i}}^n$); ▷ (calls HyperbolicMapping)

    $\mathcal{G}^p :=$ updateOverlappingGrid($\mathcal{G}^p$); ▷ (Ogen)

    $\mathbf{q}_{\mathbf{i}}^{n+1} :=$ advanceFluid($\mathbf{q}_{\mathbf{i}}^n$, $\mathcal{G}^n$, $\mathcal{G}^p$, Δt);

    $\bar{\mathbf{q}}_{\mathbf{i}}^{n+1} :=$ advanceSolid($\bar{\mathbf{q}}_{\mathbf{i}}^n$, $\mathcal{G}$, Δt);

    $\mathbf{q}_{\mathbf{i}}^{n+1} :=$ applyFluidBCs($\mathbf{q}_{\mathbf{i}}^{n+1}$, $\mathcal{G}^p$, $\mathbf{n}^T\mathbf{v}^I$, $\mathbf{n}^T\boldsymbol{\sigma}^I$);

    $\bar{\mathbf{q}}_{\mathbf{i}}^{n+1} :=$ applySolidBCs($\bar{\mathbf{q}}_{\mathbf{i}}^{n+1}$, $\mathcal{G}$, $\mathbf{n}^T\mathbf{v}^I$, $\mathbf{n}^T\boldsymbol{\sigma}^I$);

    $\mathcal{G}^{n+1} :=$ correctMovingGrids($\mathbf{q}_{\mathbf{i}}^{n+1}$, $\bar{\mathbf{q}}_{\mathbf{i}}^{n+1}$, $\mathcal{G}^p$, Δt);

    $t := t + \Delta t$; $n := n + 1$;

  end while

end procedure

Movies: Incompressible flow with rigid bodies

Traditional algorithms fail for light solids

Incompressible flow + rigid cyls

Incompressible flow + stick

drops

drop stick

FSI and Added-Mass Instabilities

Added-Mass effect:

To move a body one must also move the surrounding fluid:

$$\text{Force} = (m_{\text{b}ody} + m_{\text{added mass}}) \text{ acceleration}$$

Traditional partitioned algorithms may require many sub-time-step iterations for light solids.

Monolithic schemes are stable but can be expensive, less flexible.

Fluid –Solid interface conditions:

$$\sigma_{\text{fluid}} = \bar{\sigma}_{solid}, \qquad \text{(balance force)}$$

$$v_{\text{fluid}} = \bar{v}_{solid}. \qquad \text{(match velocity)}$$

Traditional partitioned algorithm:

1. Advance solid using stress (traction) from the fluid

$$\text{(Interface traction)} \quad \sigma^I = \sigma_{\text{fluid}}$$

2. Advance fluid using velocity from the solid,

$$\text{(Interface velocity)} \quad v^I = \bar{v}_{solid}$$

The stability problem is particularly acute for incompresssible flows (e.g. blood flow).

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The stability problem is particularly acute for incompresssible flows (e.g. blood flow).

The added-mass instability has received much attention in the literature

Approaches to partially address the added mass instability:

- Robin-Robin (mixed) boundary conditions with coefficients determined from simplified known solutions.
- interface artificial compressibility, fictitious pressure and fictitious mass.
- fixed point iterations (Aitken accelerated).
- semi-monolitic, approximate factorizations and Newton type schemes.

Causin, Gerbeau, Nobile (2005), Forster, Wall, Ramm (2007), vanBrummelen (2009) Badia, Quaini, Quarteroni (2008), Astorino, Chouly, Fernandez (2009), Degroote, Bathe, Vierendeels (2009), Guidoboni, Glowinski, Cavallini, Canic (2009) Fernandez (review, 2011), Gretarsson, Kwatra, Fedkiw (2011) Baek Karniadakis (2012) Nobile, Vergara (2012), Yu, Baek, Karniadakis (2013), Bukac, Canic, Glowinski, Tambaca, Quaini (2013), Fernandez, Mullaert, Vidrascu (2014) Fernandez, Landajuela (2014), . . .

Added Mass Partitioned (AMP) Algorithms

Overcome the added-mass instability - require no sub-iterations

In some recent papers we have shown how to over-come the added-mass instability for different FSI regimes (AMP schemes require no sub-iterations)

1. Compressible flow + compressible solids
   $\rightarrow$ Embed solution of a fluid-solid Riemann problem.
2. Compressible flow + rigid solids
   $\rightarrow$ Incorporate added-mass tensors into the rigid body equations.
3. Incompressible flow + beams/shells
   $\rightarrow$ Derive Robin pressure boundary condition by matching accelerations.
4. Incompressible flow + compresssible elastic bulk solids
   $\rightarrow$ Derive Robin pressure boundary condition from solid characteristics.

- JWB, B. Sjögreen, *A normal mode stability analysis of numerical interface conditions for fluid/structure interaction*, Commun. Comput. Phys. (2011).
- JWB, WDH, DWS, *Deforming Composite Grids for Solving Fluid Structure Problems*, JCP (2012).
- JWB, WDH, BS, *A stable FSI algorithm for light rigid bodies in compressible flow*, JCP (2013).
- JWB, WDH, DWS, *An analysis of a new stable partitioned algorithm for FSI problems. Part I: Incompressible flow and elastic solids*, JCP (2014)
- JWB, WDH, DWS, *An analysis of a new stable partitioned algorithm for FSI problems. Part II: Incompressible flow and structural shells*, JCP (2014).

AMP: Compressible flow + elastic solids

The AMP scheme for compressible flow and elastic solids is based on a impedance weight average of provisional fluid and solid values:

$$v_I = \frac{\bar{z}\bar{v} + zv}{\bar{z} + z} + \frac{\sigma - \bar{\sigma}}{\bar{z} + z},$$
$$\sigma_I = \frac{\bar{z}^{-1}\bar{\sigma} + z^{-1}\sigma}{\bar{z}^{-1} + z^{-1}} + \frac{v - \bar{v}}{\bar{z}^{-1} + z^{-1}}$$

where $\bar{z} = \bar{\rho}\bar{c}_p$ and $z = \rho a$ are the solid and fluid impedances. Derived from a fluid-solid Riemann problem.

Compressible flow + elastic solids

Shock hitting a neo-Hookean solid

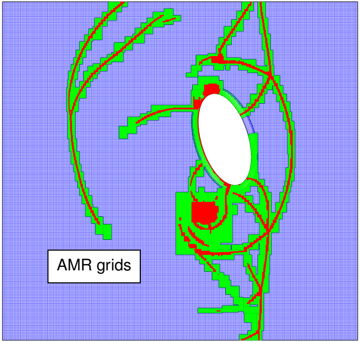
Shock hitting two elastic sticks

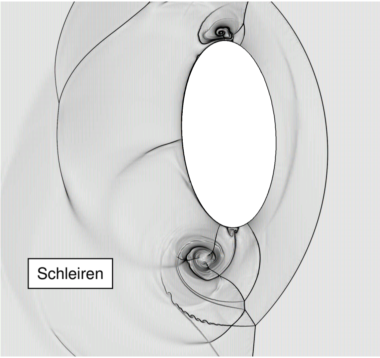
Neo-Hookean solid

deforming sticks

AMP: Compressible flow + light rigid-bodies

Approach: Analytically derive *exact* added-mass tensors for the rigid-body equations of motion. (requires certain integrals of the solution on the body surface).





Shock hitting an ellipse of *zero* mass.

- J.W. Banks, WDH, Sjögreen, *A stable FSI algorithm for light rigid bodies in compressible flow*, JCP (2013).

AMP: Compressible flow + light rigid-bodies

The Newton-Euler equations of rigid body motion are

$$m_b \dot{\mathbf{v}}_b = \mathcal{F}, \qquad (\mathcal{F} : \text{force}),$$
$$M_I \dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times (M_I \boldsymbol{\omega}) + \mathcal{T}, \qquad (\mathcal{T} : \text{Torque}),$$

$\mathbf{v}_b, \boldsymbol{\omega} \in \mathbb{R}^3$: linear and angular velocity of the centre of mass,
$m_b \in \mathbb{R}, \quad M_I \in \mathbb{R}^{3\times 3}$: mass and moment of inertia matrix.

From the solution to a *fluid-solid Riemann problem*, we analytically expose the implicit dependence of $\mathcal{F}$ and $\mathcal{T}$ on $\mathbf{v}_b$ and ω, e.g.,

$$\mathcal{F} = \int_{\partial B} -p\,\mathbf{n}\,dS = -A^{vv}\mathbf{v}_b - A^{v\omega}\omega + \widetilde{\mathcal{F}}.$$

The matrices A^{ij} are the *added-mass* tensors; for example,

$$A^{vv} = \int_{\partial B} z_f \mathbf{n}\mathbf{n}^T \, ds,$$

where $z_f = \rho_f c_f$ is the fluid impedance, and $\mathbf{n} \in \mathbb{R}^3$ is the unit normal.

AMP: Compressible flow + light rigid-bodies

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Shock hitting a rigid ellipse of *zero* mass

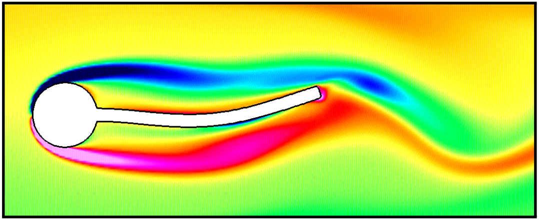
Impossible to solve with the traditional partitioned scheme

zero mass ellipse

FSI for Incompressible Flows

Traditional partitioned schemes have historically suffered from added mass instabilities for light solids

Incompressible flows are particularly difficult due to the infinite speed of sound.



Flapping beam (requires tens of sub-iterations per time step with traditional scheme).

1. Traditional FSI partitioned schemes often fail for light solids and require multiple sub-iterations per time-step.
2. The problem originates with the *added mass* effect – to move a body one must also move the surrounding fluid.
3. Incompressible flows are important (e.g. blood flow in a vein, flapping flag, underwater structures) but are particularly difficult.

Incompressible flow with Beams

Using traditional scheme and many (20-40) sub time-step iterations.

Incompressible flow + two beams

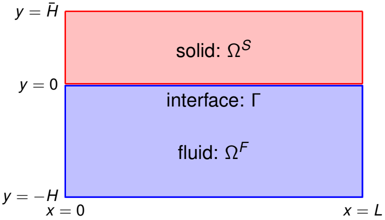
two beams

cyl beam

Incompressible flow + flexible beam

Incompressible Stokes fluid + elastic bulk solid.

For analysis and evaluation we consider an FSI model problem.



Incompressible Stokes fluid and compressible elastic bulk solid equations:

Fluid: $$\begin{cases} \rho \dfrac{\partial \mathbf{v}}{\partial t} + \nabla p = \mu \Delta \mathbf{v}, & \mathbf{x} \in \Omega^F, \\ \nabla \cdot \mathbf{v} = 0, & \mathbf{x} \in \Omega^F, \\ \mathbf{v}(x,-H,t) = 0, & \end{cases}$$

Solid: $$\begin{cases} \bar\rho \dfrac{\partial^2 \bar{\mathbf{u}}}{\partial t^2} = (\bar\lambda + \bar\mu)\nabla(\nabla\cdot\bar{\mathbf{u}}) + \bar\mu \Delta \bar{\mathbf{u}}, & \mathbf{x} \in \Omega^S, \\ \bar{\mathbf{u}}(x,\bar H,t) = 0, & \end{cases}$$

Interface: $$\mathbf{v} = \frac{\partial \bar{\mathbf{u}}}{\partial t},$$

$$\mu\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) = \bar\mu\left(\frac{\partial \bar u_1}{\partial y} + \frac{\partial \bar u_2}{\partial x}\right), \quad -p + 2\mu \frac{\partial v_2}{\partial y} = \bar\lambda \nabla\cdot\bar{\mathbf{u}} + 2\bar\mu \frac{\partial \bar u_2}{\partial y}, \quad \mathbf{x}\in\Gamma.$$

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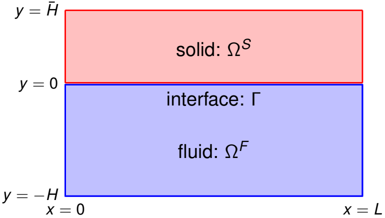

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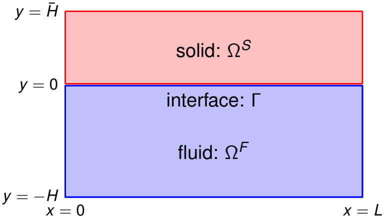

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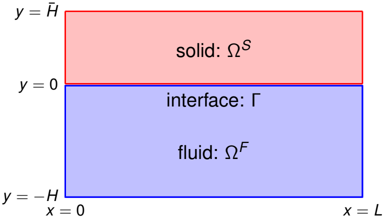

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AMP schemes for INS + bulk solids

Robin condition is derived from outgoing solid characteristic.

Key ingredient I: Given predicted values for the solid, use the out-going solid characteristic variables, to define the fluid interface conditions

$$\mathbf{n}^T\boldsymbol{\sigma}\mathbf{n} + \bar{z}_p\mathbf{n}^T\mathbf{v} = \mathbf{n}^T\bar{\boldsymbol{\sigma}}^{(p)}\mathbf{n} + \bar{z}_p\mathbf{n}^T\bar{\mathbf{v}}^{(p)} \equiv \mathcal{B}(\bar{\boldsymbol{\sigma}}^{(p)}, \bar{\mathbf{v}}^{(p)}), \qquad \mathbf{x} \in \Gamma$$

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$\bar{z}_p = \bar{\rho}\bar{c}_p$ and $\bar{z}_s = \bar{\rho}\bar{c}_s$ are the solid impedances , OR since $\boldsymbol{\sigma} = -pI + \boldsymbol{\tau}$,

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Key ingredient II: Matching *accelerations* instead of *velocities* gives a Robin condition for the fluid pressure

$$\boxed{-p - \frac{\bar{z}_p\Delta t}{\rho}\frac{\partial p}{\partial n}} + \mathbf{n}^T\boldsymbol{\tau}\mathbf{n} + \frac{\mu\bar{z}_p\Delta t}{\rho}\mathbf{n}^T(\Delta\mathbf{v}) = \mathbf{n}^T\bar{\boldsymbol{\sigma}}^{(p)}\mathbf{n} + \bar{z}_p\Delta t\,\mathbf{n}^T\frac{\partial\bar{\mathbf{v}}^{(p)}}{\partial t}, \qquad \mathbf{x} \in \Gamma$$

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AMP INS-BULK: more details

Incompressible Navier-Stokes equations:

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) + \nabla p = \mu\Delta\mathbf{v}, \quad \mathbf{x}\in\Omega^F, \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0, \quad \mathbf{x}\in\Omega^F, \tag{2}$$

We like to solve the the velocity pressure form,

$$\rho\Big(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\Big) + \nabla p = \mu\Delta\mathbf{v}, \quad \mathbf{x}\in\Omega^F, \tag{3}$$

$$\Delta p = -\rho\nabla\mathbf{v} : \nabla\mathbf{v}, \quad \mathbf{x}\in\Omega^F, \tag{4}$$

with additional boundary condition $\nabla\cdot\mathbf{v} = 0$.

AMP INS-BULK: more details

The INS equations are solved with a fractional-step scheme

Fractional-step scheme:

Stage I: advance the fluid using past time values of the pressure:

$$\rho \frac{\mathbf{v}^p - \mathbf{v}^n}{\Delta t} = \frac{\mu}{2}\left(\Delta \mathbf{v}^p + \Delta \mathbf{v}^n\right) + \frac{3}{2}\mathbf{F}^n - \frac{1}{2}\mathbf{F}^{n-1}, \quad (\mathbf{F}^n = -\rho(\mathbf{v}^n \cdot \nabla)\mathbf{v}^n - \nabla p^n)$$

Stage II: solve for the pressure using the AMP Robin (mixed) BC:

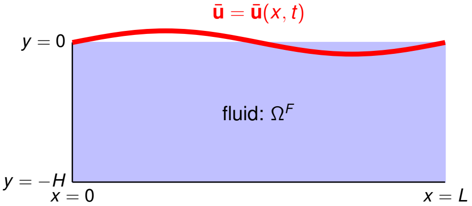
$$\Delta p^p = -\rho \nabla \mathbf{v}^p : \nabla \mathbf{v}^p,$$
$$-p - \frac{\bar{z}_p \Delta t}{\rho}\frac{\partial p}{\partial n} = G(\mathbf{v}^p, \bar{\mathbf{v}}^{(p)}, \bar{\boldsymbol{\sigma}}^{(p)})$$

Optionally correct,

$$\rho \frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} = \frac{\mu}{2}\left(\Delta \mathbf{v}^{n+1} + \Delta \mathbf{v}^n\right) + \frac{1}{2}\mathbf{F}^p + \frac{1}{2}\mathbf{F}^n,$$

with another pressure solve.
This scheme can be made fully second-order accurate in the max-norm.

Incompressible Stokes fluid + beam/shell.



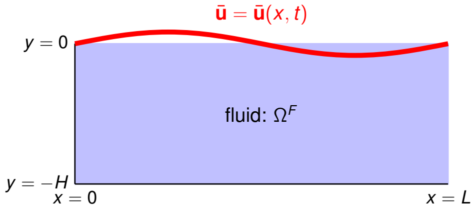
Incompressible Stokes fluid and beam equations:

Fluid: $\begin{cases} \rho \dfrac{\partial \mathbf{v}}{\partial t} + \nabla p = \mu \Delta \mathbf{v}, & \mathbf{x} \in \Omega^F, \\ \nabla \cdot \mathbf{v} = 0, & \mathbf{x} \in \Omega^F, \\ \mathbf{v}(x,-H,t) = 0, & \end{cases}$ Solid: $\left\{ \bar{\rho} h_s \bar{\mathbf{v}}_t = \bar{\mathbf{L}}(\bar{\mathbf{u}}) - \sigma \mathbf{n}, \quad \mathbf{x} \in \Gamma, \right.$

Interface: $\mathbf{v} = \dfrac{\partial \bar{\mathbf{u}}}{\partial t}$

Example beam operator: $\bar{\mathbf{L}}(\bar{\mathbf{u}}) = -EI\, \bar{\mathbf{u}}_{xxxx}$.

Incompressible Stokes fluid + beam/shell.



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AMP schemes for INS + beams or shells

Robin condition derives from matching *accelerations*

Beam equation for solid

$$\bar{\rho}\bar{h}\,\bar{\mathbf{v}}_t = \bar{\mathbf{L}}(\bar{\mathbf{u}}) - \boldsymbol{\sigma}\mathbf{n}, \qquad \mathbf{x} \in \Gamma,$$

Matching *accelerations* $\mathbf{v}_t = \bar{\mathbf{v}}_t$ implies

$$\bar{\rho}\bar{h}\,\mathbf{v}_t = \bar{\mathbf{L}}(\bar{\mathbf{u}}) - \boldsymbol{\sigma}\mathbf{n}, \qquad \mathbf{x} \in \Gamma,$$

Using $\rho\mathbf{v}_t = \nabla\cdot\boldsymbol{\sigma}$ and given a predicted value for the beam $\bar{\mathbf{u}}^{(p)}$, the Robin conditions for the fluid are

$$\boldsymbol{\sigma}\mathbf{n} + \frac{\bar{\rho}\bar{h}}{\rho}\nabla\cdot\boldsymbol{\sigma} = \bar{\mathbf{L}}\big(\bar{\mathbf{u}}^{(p)}\big), \qquad \mathbf{x} \in \Gamma,$$

Which includes a Robin condition for the pressure

$$\boxed{p + \frac{\bar{\rho}\bar{h}}{\rho}\frac{\partial p}{\partial n}} = \mathbf{n}^T\boldsymbol{\tau}\mathbf{n} + \frac{\mu\bar{\rho}\bar{h}}{\rho}\mathbf{n}^T\Delta\mathbf{v} - \mathbf{n}^T\bar{\mathbf{L}}\big(\bar{\mathbf{u}}^{(p)}\big), \qquad \mathbf{x} \in \Gamma$$

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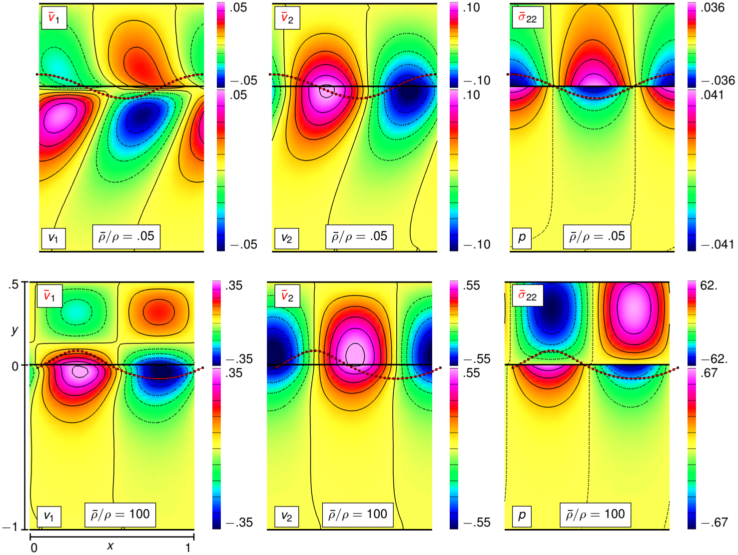
Traveling wave solutions: Stokes + Elastic Solid

Computed solutions for a light and heavy solid match analytic solutions.



Numerical results demonstrate stability and second-order accuracy of the AMP schemes

Confirming normal-mode analysis.

MP-VE, traveling wave, viscous fluid, $\mu = .02$, heavy elastic solid, $\bar{\rho}/\rho = 10^3$										
h_j	$E_j^{(p)}$	r	$E_j^{(\mathbf{v})}$	r	$E_j^{(\bar{\mathbf{u}})}$	r	$E_j^{(\bar{\mathbf{v}})}$	r	$E_j^{(\bar{\boldsymbol{\sigma}})}$	r
1/20	1.2e-2		1.9e-2		2.4e-3		1.6e-2		3.5e1	
1/40	2.9e-3	4.1	3.7e-3	5.1	4.5e-4	5.4	3.1e-3	5.2	9.1e0	3.9
1/80	6.5e-4	4.5	6.0e-4	6.1	8.3e-5	5.4	6.0e-4	5.1	2.5e0	3.6
1/160	1.5e-4	4.3	1.3e-4	4.8	1.6e-5	5.0	1.2e-4	4.8	6.8e-1	3.7
rate	2.18		2.21		2.03		2.03		1.95	

MP-VE, traveling wave, viscous fluid, $\mu = .005$, very light elastic solid, $\bar{\rho}/\rho = 10^{-3}$										
h_j	$E_j^{(p)}$	r	$E_j^{(\mathbf{v})}$	r	$E_j^{(\bar{\mathbf{u}})}$	r	$E_j^{(\bar{\mathbf{v}})}$	r	$E_j^{(\bar{\boldsymbol{\sigma}})}$	r
1/20	2.1e-5		3.2e-4		8.0e-4		2.4e-3		1.3e-5	
1/40	4.6e-6	4.6	9.2e-5	3.4	1.6e-4	4.9	5.8e-4	4.2	3.4e-6	3.9
1/80	9.8e-7	4.7	2.3e-5	4.0	2.7e-5	6.1	10.0e-5	5.8	1.1e-6	3.2
1/160	2.2e-7	4.5	5.7e-6	4.0	4.3e-6	6.3	2.2e-5	4.6	2.9e-7	3.6
rate	2.21		1.93		2.53		2.29		1.82	

Traveling wave solution for a viscous incompressible fluid and elastic solid (MP-VE).

Bulk solids: Traditional scheme is always unstable!
(on sufficiently fine grids, without sub-iterations)

Analysis shows: The Traditional Partitioned (TP) algorithm is formally *unconditionally unstable* (on a fine enough grid).

Theorem

The TP algorithm is stable if and only if $\Delta t \le \frac{2}{\bar{c}_p}\left(\Delta y - \frac{\rho H}{\bar{\rho}}\right)$.

2D computations confirm the theory:

MP-VE, traveling wave, TP algorithm				
δy	$\bar{\rho}/\rho = 800$	$\bar{\rho}/\rho = 400$	$\bar{\rho}/\rho = 200$	$\bar{\rho}/\rho = 100$
1/20	stable	stable	stable	stable
1/40	stable	stable	stable	unstable
1/80	stable	stable	unstable	unstable
1/160	stable	unstable	unstable	unstable
1/320	unstable	unstable	unstable	unstable

Incompressible flow with Beams - AMP results

Incompressible flow + two beams ($\bar{\rho} = \rho$)

two beams

Artery pressure pulse

Model of an artery

Summary

1. Overture can be used to solve FSI problems in a variety of flow regimes.
2. Traditional partitioned schemes for FSI suffer from an *added-mass* instability for light solids.
3. Traditional schemes can be stabilized by using multiple sub-iterations per time step, or through the use of Robin (mixed) interface conditions (which still generally require sub-iterations).
4. We have developed AMP schemes that over-come the added-mass instability for a variety of regimes and *require no sub-iterations*.
5. AMP schemes for incompressible flows and compressible elastic bulk solids or elastic shells/beams were described and shown to be stable without iterations, even for light solids.

The Overture software is freely available from `overtureFramework.org`